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Evolution of unstable shear layers in a rotating fluid

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The time development of non-axisymmetric disturbances on a shear layer in a uniformly rotating fluid is studied theoretically. It is assumed that the Rossby number, Ekman number, shear-layer length scale, and initial disturbance amplitude are small, and that disturbances grow if vorticity transfer from the shear flow exceeds Ekman-layer vorticity dissipation, a mechanism investigated by Busse (1968). Expressions for the ultimate instability amplitude are determined for specified relations among the small parameters, using the volume-integrated energy equation, the shape assumption, and approximations for the spatial dependence of disturbances. Disturbance growth is limited by modification of the initially-unstable shear-layer profile, and the eventual amplitude is shown to be fairly insensitive to the specific form of the profile. Using data from observations by Hide & Titman, the predicted maximum velocity amplitude of the non-axisymmetric motions for their experiments is approximately one-quarter of the velocity of the shear flow at the point of maximum gradient.

1. Introduction

The Taylor–Proudman theorem determines the salient characteristic of small, nearly steady motions that are deviations from uniform rotation in a slightly viscous incompressible fluid. The theorem's requirement of two-dimensionality is responsible for the frequent occurrence of thin shear layers parallel to the rotation axis (Greenspan 1968). The shear layers have been investigated experimentally, and under certain flow conditions they are susceptible to instability. An interesting series of experiments by Hide & Titman (1967) demonstrated the breakdown of an initially axisymmetric shear layer into a non-axisymmetric flow pattern. The shear layer was generated inside a rotating fluid-filled cylindrical tank by differentially rotating a thin coaxial disk. The instability possesses some striking features, among them that the non-axisymmetric flow itself shows little variation in the direction of the rotation axis. If the disk rotates in the same direction as the mean rotation, the fluid motions in the non-axisymmetric flow consist of m vortex-like circulation cells ($m \leqslant 6$) arranged around the rotation axis, the pattern of cells moving relative to both disk and tank. Hide & Titman determined the condition for onset of the nonaxisymmetric regime as a relation between the Ekman number E and the Rossby number R_0 (see §2 for definitions), and they also gave results for the number m and for the pattern travel speed at onset.

An important paper by Busse (1968) argued that the shear-layer instability exemplified by Hide & Titman's experiments was produced by the same vorticity-convection mechanism that is responsible for instability of inviscid plane parallel shear flow. Development of the instability is retarded by dissipation in the Ekman boundary layers, an effect which is readily incorporated into the governing equations using boundary-layer compatibility conditions. The linear instability equation differs from the familiar Rayleigh equation only because of the cylindrical geometry. Properties of a generalized version of the instability equation which admits three-dimensional disturbances had been extensively reported earlier by Howard & Gupta (1962) and Michalke & Timme (1967).

In view of the regularity and persistence of the non-axisymmetric flow pattern as indicated by Hide & Titman's results, it is natural to seek to understand the mechanism that limits the growth of the instability. A study of the evolution of the shear-layer instability is germane as an indication of the effect of rotation on the initial stage of transition to turbulence. The theory to follow is particularly aimed at revealing the parametric dependence of the ultimate instability amplitude, and at producing predictions which may be tested by suitable experiments. The theory was in fact conceived in conjunction with a series of experiments at another institution, in which re-examination of Hide & Titman's results was planned, but these experiments were terminated before sufficiently detailed measurements were made. An additional motivation is the possible operation of a similar instability in the creation and development of small but deadly 'suction vortices' near the radius of maximum azimuthal velocity of a tornado (Fujita 1971).

In the model we adopt, instability arises in the manner described by Busse, when energy transfer from the axisymmetric flow by the perturbation Reynolds stress overcomes the perturbation energy loss to the Ekman layers. In spite of the appeal of this mechanism, the quantitative evidence for its operation in the Hide & Titman experiments is not compelling (see §6 of Busse's paper). The issue is unlikely to be resolved without additional experiments, however, and following theory is grounded in the assumption that Busse's ideas adequately describe instability onset. As the non-axisymmetric motion evolves, it is capable of modifying the axisymmetric flow through quadratic interaction terms of the fluid equations. The distortion of the axisymmetric flow simultaneously affects the development of the unstable perturbations, and a primary consequence of our model is that the modified axisymmetric flow *retards* the energy transfer to the perturbations. The modification of the axisymmetric flow is thus the mechanism that limits perturbation growth, a process described lucidly by Stuart (1958) for other stability problems.

The axisymmetric flow on which the instability develops is generated by differential rotation of one of the horizontal boundaries. We assume that the wall motion differs from uniform rotation only in a region of small radial extent, characterized by a dimensionless length ϵ. Consequently, all the important energy transfer contributions in the instability occur in a thin shear layer, the thickness of which depends on the relation between the parameters ϵ and E. Our results are appropriate for the case when ϵ is small compared with one but sufficiently large that lateral dissipation, i.e. that due to viscous terms with radial and azimuthal derivatives, does not enter the lowest-order dynamics of the shear layer.

The requirement on ϵ (specifically $\epsilon \gg E^{\frac{1}{4}}$) is a significant restriction, since it eliminates dissipation outside the Ekman layers as an effective mechanism for delaying the onset and limiting the growth of the instability. Obviously, lateral dissipation must be influential in stabilizing the large-m perturbations at onset and in determining the detailed structure of the shear layer; for the case of discontinuous wall motion $\epsilon = 0$, the shear-layer dynamics rely entirely on the interaction of Coriolis and lateral shear forces. On the other hand, it is well known (see remarks by Busse) that dynamical theories which incorporate dissipation only through the Ekman layers give impressively accurate predictions for the actual behaviour of rotating fluids. As further justification for our assumption, calculations for the finite amplitude of shear-layer instabilities in non-rotating fluids (Schade 1964; Michalke 1965; Stuart 1967) demonstrate that in those problems the limiting amplitude of the unstable flow is actually independent of viscosity, and Stuart and Schade report some experimental agreement with their conclusions. If indeed the *details* of the shear-layer structure are not crucial in fixing the limiting perturbation amplitude, then it is reasonable and useful to examine the mathematically simpler finite-amplitude stability problem without the mechanism of lateral dissipation.

The method used here for the calculation of the eventual instability amplitude was developed by Stuart (1958) and centres on the volume-integrated perturbation energy equation, supplemented by an equation for the azimuthally-averaged flow and the 'shape assumption' for the spatial dependence of the perturbation. The so-called Stuart–Watson (1960) procedure, and other more recent methods based on their formalism, are not applicable to the present problem. The reason is that there are no known everywhere differentiable, neutrally-stable solutions of the linear stability equation (2.21). The available piecewise-smooth solutions are unsuitable, because the nonlinear terms intensify the singularities in higher approximations to the solutions. Use of even the more primitive method in this paper depends crucially on the azimuthally-averaged equation (2.13) which governs the subsequent evolution of the initially unstable azimuthal velocity. The corresponding equation for inviscid *non-rotating* flow requires that the perturbation Reynolds stress vanish at equilibrium. In the absence of rotation, it is therefore necessary to use a method of the Stuart–Watson type to investigate the amplitude limitation mechanism.

The shape assumption and amplitude calculations require approximations for the spatial form of the unstable eigenfunction and for its growth rate close to onset conditions. These are obtained by adapting the long-wave approximation procedure of Drazin & Howard (1962) to the present problem. The thinness of the shear layer produces a physical situation for instability waves of moderate m which is analogous to that experienced by long waves on a moderately thick shear layer. The solution is found by expanding in the two parameters ϵ and $E^{\frac{1}{2}}$ while exploiting the assumption $E^{\frac{1}{4}} \ll \epsilon$. The parameter R_0 is assumed $O(E^{\frac{1}{2}})$ in accord with Busse's theory; Hide & Titman's experiments predict the exponent $\frac{4}{7}$ instead of $\frac{1}{2}$, although their parameter definitions differ slightly from Busse's.

The validity of our particular expansion procedure requires the characteristic amplitude scale δ of the initial perturbation to satisfy $\delta = O(\epsilon)$, but such a restriction is not unexpected and is typical of perturbation schemes for weakly nonlinear equations.

In §2 the equations for the averaged flow and linear stability problem are derived, while the former is simplified, solved, and combined with the volume-integrated perturbation energy equation in §3. Approximations for the spatial form of the unstable perturbation and its growth rate are obtained in §4. The equation governing the limiting amplitude of the perturbation is derived in §5, and the results are summarized and discussed in §6.

2. Formulation

We consider a homogeneous fluid of unit density and kinematic viscosity ν, contained between two rigid parallel infinite disks which are separated by a distance H and are rotating with angular velocity Ω about their common axis. The fluid pressure p and velocity u are measured relative to a co-rotating cylindrical co-ordinate system, and the velocity components in the radial (r), azimuthal (θ), and vertical (z) directions are u, v, and w, respectively. We non-dimensionalize the variables $\mathbf{r}$, $\mathbf{u}$, p, and time t by H, ωH^2, and $\Omega^{-1}T^{-1}$, where ω is a temporarily unspecified characteristic angular velocity magnitude and T is a dimensionless scaling factor. Then $\mathbf{u}$ and p satisfy the Navier–Stokes equations, written here in dimensionless variables,

$$T\mathbf{u}_t + R_0(\mathbf{u}.\nabla)\mathbf{u} + 2\hat{\mathbf{k}}\times\mathbf{u} + \nabla p = E\nabla^2\mathbf{u}, \tag{2.1}$$

$$\nabla.\mathbf{u} = 0, \tag{2.2}$$

wherein the parameters $R_0 = \omega\Omega^{-1}$ (Rossby number) and $E = \nu(rH^2)^{-1}$ (Ekman number).

We are interested in the stability of a class of flows which are approximate solutions of (2.1)–(2.2) for certain ranges of small values for R_0 and E. The basic flow is generated by differential rotation of the upper disk, so that no-slip boundary conditions have the dimensionless form

$$\left.\begin{aligned} \mathbf{u} &= 0 \quad \text{at} \quad z = 0, \\ \mathbf{u} = V_w(r)\,\hat{\boldsymbol{\theta}} &= r\Omega_w(r)\,\hat{\boldsymbol{\theta}} \quad \text{at} \quad z = 1. \end{aligned}\right\} \tag{2.3}$$

The dimensionless wall angular velocity Ω_w is normalized so that its maximum value is unity, choosing the scale ω to be the actual maximum value of the dimensional wall angular velocity. The function Ω_w is the step function $H(a-r)$ in the model of Stewartson (1957), and a step function is also an appropriate representation of the experimental configuration of Hide & Titman (1967). There are experimental difficulties in producing any disk motion other than a step function or a sum of step functions. Nevertheless, an important element of the present model is the assumption that the wall angular velocity can adopt a somewhat different form which is specified in §4. Roughly speaking, Ω_w is regarded as a function of $(r-a)/\epsilon$, where ϵ is small compared with one but large compared with $E^{\frac{1}{4}}$, and Ω_w is essentially constant outside a neighbourhood of $r = a$.

We suppose the velocity and pressure fields are written as

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{U}(r,z,t) + \delta\mathbf{u}'(r,\theta,z,t), \\ p &= P(r,z,t) + \delta p'(r,\theta,z,t), \end{aligned}\right\} \tag{2.4}$$

where $$\mathbf{U}(r,z,t) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{u}(r,\theta,z,t)\,d\theta \equiv \overline{\mathbf{u}(r,\theta,z,t)}$$

is the azimuthally-averaged velocity field. The parameter δ is the ratio of the characteristic amplitude of the non-axisymmetric perturbation fields to the axisymmetric averaged fields, and is presumed small compared with one. Using (2.4) in (2.1)–(2.2) and hereafter dropping the primes on the perturbation fields, the governing equations are

$$T\mathbf{U}_t + R_0[\mathbf{U}.\nabla\mathbf{U} + \delta^2\overline{\mathbf{u}.\nabla\mathbf{u}}] + 2\hat{\mathbf{k}}\times\mathbf{U} + \nabla P = E\nabla^2\mathbf{U}, \tag{2.5}$$

$$r^{-1}(rU)_r + W_z = 0, \tag{2.6}$$

$$T\mathbf{u}_t + R_0[\mathbf{U}.\nabla\mathbf{u} + \mathbf{u}.\nabla\mathbf{U}] + R_0\delta[\mathbf{u}.\nabla\mathbf{u} - \overline{\mathbf{u}.\nabla\mathbf{u}}] + 2\hat{\mathbf{k}}\times\mathbf{u} + \nabla p = E\nabla^2\mathbf{u}, \tag{2.7}$$

$$\nabla.\mathbf{u} = 0. \tag{2.8}$$

In the regime of small E, Ekman boundary layers are formed on the disk surfaces, and as is well known conditions (2.3) can be replaced by compatibility conditions satisfied by the velocity fields outside the Ekman layers (Greenspan 1968, p. 92),

$$\begin{aligned} W &= \tfrac{1}{2}E^{\frac{1}{2}}r^{-1}(rV)_r \quad \text{at} \quad z = 0, \\ W &= -\tfrac{1}{2}E^{\frac{1}{2}}r^{-1}(rV - rV_w)_r \quad \text{at} \quad z = 1, \end{aligned} \tag{2.9}$$

$$w = \pm\tfrac{1}{2}E^{\frac{1}{2}}r^{-1}[(rv)_r - u_\theta] \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{2.10}$$

The velocity is of course required to be bounded for large r.

Consider first the equations for the averaged flow. Because the boundary conditions are time-independent, initial conditions can be selected so that time variation of the averaged flow arises only from the third term in (2.5), and we can then require T to be small as well as R_0 and E. From (2.5) and (2.6) we readily verify that the largest non-vanishing components of velocity and pressure have the properties

$$U_z = 0, \quad V_z = 0, \quad P_z = 0, \quad W_{zz} = 0, \tag{2.11}$$

and, using (2.9) and (2.6),

$$\left.\begin{aligned} W &= E^{\frac{1}{2}}r^{-1}(r[\tfrac{1}{2}V + z(\tfrac{1}{2}V_w - V)])_r, \\ U &= E^{\frac{1}{2}}(V - \tfrac{1}{2}V_w). \end{aligned}\right\} \tag{2.12}$$

Equations (2.11) and (2.12) represent the information available in the largest terms of (2.5) and (2.6). To obtain an equation governing the $O(1)$ component V, we use (2.11) and (2.12) in the azimuthal component of (2.5) and retain smaller terms:

$$\begin{aligned} TV_t + R_0E^{\frac{1}{2}}(V - \tfrac{1}{2}V_w)\,r^{-1}(rV)_r + R_0\delta^2[\overline{ur^{-1}(rv)_r} + \tfrac{1}{2}r^{-1}\overline{(v^2)}_\theta \\ + \overline{wv_z}] + 2E^{\frac{1}{2}}(V - \tfrac{1}{2}V_w) = E(r^{-1}(rV)_r)_r. \end{aligned}$$

Following Busse, we define $\beta = R_0 E^{\frac{1}{2}}$ and assume $\beta = O(1)$ as $R_0 \to 0$, so this equation becomes

$$2V = V_w - \beta\delta^2[\overline{ur^{-1}(rv)_r} + \overline{wv_z}] + E^{\frac{1}{2}}(r^{-1}(rV)_r)_r - R_0(V - \tfrac{1}{2}V_w)r^{-1}(rV)_r - TE^{-\frac{1}{2}}V_t. \quad (2.13)$$

In view of Busse's paper, we shall only briefly consider the equations for the perturbation fields. With the asymptotic relation $R_0 = O(E^{\frac{1}{2}})$ it is appropriate to expand the perturbation fields as

$$\mathbf{u} = \mathbf{u}^0 + R_0\mathbf{u}^1 + \dots, \quad p = p^0 + R_0 p^1 + \dots,$$

and to investigate instability with $T = R_0$. Then (2.7) and (2.8), along with $\mathbf{U} = V(r)\hat{\boldsymbol{\theta}} + O(E^{\frac{1}{2}})$, give

$$\mathbf{u}^0 = \tfrac{1}{2}\hat{\mathbf{k}} \times \nabla p^0, \quad p_z^0 = 0, \quad (2.14)$$

$$\mathbf{u}_t^0 + Vr^{-1}\mathbf{u}_\theta^0 + [-2r^{-1}Vv^0\hat{\mathbf{r}} + r^{-1}(rV)_r u^0\hat{\boldsymbol{\theta}}] + 2\hat{\mathbf{k}} \times \mathbf{u}^1 + \nabla p^1 = 0, \quad (2.15)$$

$$r^{-1}(ru^1)_r + r^{-1}v_\theta^1 + w_z^1 = 0, \quad (2.16)$$

and, with (2.14), (2.10) becomes

$$w^1 = \pm\tfrac{1}{4}\beta^{-1}\nabla^2 p^0 \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}. \quad (2.17)$$

Combining (2.14)–(2.17), we obtain Busse's equation

$$\nabla^2 p_t^0 + r^{-1}V\nabla^2 p_\theta^0 - r^{-1}p_\theta^0(r^{-1}(rV)_r)_r + 2\beta^{-1}\nabla^2 p^0 = 0. \quad (2.18)$$

Wavelike solutions of (2.18) have the form

$$p^0(r,\theta,t) = \mathrm{Re}\{\chi(r)\exp\{i(m\theta - \omega t)\}\}, \quad (2.19)$$

and with the definitions

$$mc = \omega + 2i\beta^{-1}, \quad \Omega(r) = r^{-1}V(r), \quad \phi(r) = \Omega(r) - c, \quad (2.20)$$

(2.18) becomes

$$\phi(\chi'' + r^{-1}\chi' - m^2r^{-2}\chi) - (\phi'' + 3r^{-1}\phi')\chi = 0. \quad (2.21)$$

Boundedness conditions must be satisfied by solutions of (2.21) at $r = 0$ and as $r \to \infty$. Our initial perturbation is taken in the form (2.19) for some m and corresponding $\chi(r)$. The parameter δ may therefore be defined as equal to, or (depending on the normalization) proportional to, the pressure maximum.

It is worth remarking that two possibly important effects in (2.7) are omitted in (2.18). The nonlinear terms in (2.7) generate harmonic components of the wave disturbance (2.19), and their neglect cannot be justified for sufficiently large perturbation amplitude. The neglect of the right side of (2.7) requires that spatial gradients of the perturbation are not too large. If the perturbation varies on a length scale as small as $O(E^{\frac{1}{4}})$, then dissipation outside the Ekman layers is formally comparable to the convection and Ekman dissipation effects retained in (2.18).

3. Energy equation for the disturbance

A disturbance in the form (2.19) will grow in time if $\operatorname{Im}\omega$ is negative. Busse (1968) found the dispersion relation, and demonstrated instability for certain angular velocities $\Omega(r)$ which are combinations of constants and terms proportional to r^{-2}, for which the second group of terms in (2.21) vanish. Michalke & Timme (1967) analysed other models similarly, and in addition verified instability for a profile $\Omega(r)$ for which (2.21) does not reduce to a simple equation for χ.

With the possibility of instability clearly established, for some m and β at least, we seek to understand the mechanism which limits the growth of the perturbation by using a procedure devised by Stuart (1958). The energy equation for the disturbance is found by dotting (2.15) with $\mathbf{u}^0$ and integrating over the flow domain,

$$\int_0^1 dz \int_0^\infty r\,dr\{\tfrac{1}{2}[\overline{(u^0)^2}+\overline{(v^0)^2}]_t+\overline{u^0v^0}r\Omega'+2(\overline{v^0u^1}-\overline{u^0v^1})+\overline{\mathbf{u}^0.\nabla p^1}\}=0. \quad (3.1)$$

Using (2.14), (2.16), (2.17) and the boundedness conditions in r, (3.1) becomes

$$\int_0^\infty r\,dr\{\tfrac{1}{2}[\overline{(u^0)^2}+\overline{(v^0)^2}]_t+2\beta^{-1}[\overline{(u^0)^2}+\overline{(v^0)^2}]+r\Omega'\overline{u^0v^0}\}=0. \quad (3.2)$$

Equation (3.2) expresses the rate of change of perturbation energy in terms of dissipation due to the Ekman layers and work by the Reynolds stress on the averaged angular velocity shear.

An expression for the evaluation of the averaged angular velocity is provided by (2.13). Now in accord with our assumption that near $r=a$ derivatives of Ω_w are $O(\epsilon^{-1})$ with $\epsilon^{-1} \ll E^{-\frac{1}{4}}$, we expect that derivatives of V are $O(\epsilon^{-1})$ as well, so that the third term on the right side of (2.13) is small compared with the terms $2V$ and V_w. We note that an expression for V can be obtained from (2.13) even with the inclusion of this term (i.e. with $\epsilon = O(E^{\frac{1}{4}})$), but we do not pursue this case. In addition, the fourth term of (2.13), which represents radial convection of the averaged shear, is small, $O(R_0\epsilon^{-1})$, under the conditions already imposed. We retain the second terms on the right side of (2.13) in the form

$$-\beta\delta^2\overline{u^0r^{-1}(rv^0)_r},$$

which formally imposes the requirements

$$O(u^0v^0_r).\delta^2 \gg \max(E^{\frac{1}{2}}\epsilon^{-2},\, R_0\epsilon^{-1}) = (E^{\frac{1}{4}}\epsilon^{-1})^2, \quad (3.3)$$

so that the orders of the terms retained are larger than those omitted.

In the original applications of his method, Stuart (1958) argued that terms analogous to the last term in (2.13) could be ignored in computing the equilibrium disturbance amplitude. We make the conservative estimate $T=R_0$ to examine its effect. Then the solution of the simplified version of (2.13) is

$$V(r,t)=\tfrac{1}{2}V_w(r)-\delta^2\exp\{-2t\beta^{-1}\}\int_0^t \overline{u^0r^{-1}(rv^0)_r}\exp\{2s\beta^{-1}\}\,ds, \quad (3.4)$$

and the solution for any initial condition other than $V(r,0)=\tfrac{1}{2}V_w$ differs from (3.4) by a transient exponential.

We can determine the time evolution and limiting amplitude of the perturbation by using (3.4) in (3.2), knowing expressions for the perturbation velocities. Following Stuart, we make the 'shape assumption', in which the perturbation velocities in the unstable or supercritical parameter regime are assumed to have the same spatial forms as those of the critically unstable motions. This approximation is of crucial importance. Its justification is possible only *a posteriori*, but more widely-used methods, such as the harmonic-generation Stuart–Watson scheme, rely on the ability of the critically unstable eigenfunctions closely to approximate the slightly supercritical ones. Moreover, the spatial integrals used along with the shape assumption could be expected to mitigate the effect of deviations from true supercritical solutions.

Consistent with (2.14) and (2.19), we therefore assume the forms

$$\left.\begin{aligned} p^0 &= \mathrm{Re}\{A(t)\,\chi(r)\exp\{i(m\theta-\omega_R t)\}\},\\ u^0 &= \mathrm{Re}\left\{A\left(-\frac{mi}{2r}\right)\chi\exp\{i(m\theta-\omega_R t)\}\right\},\\ v^0 &= \mathrm{Re}\{A\,\tfrac{1}{2}\chi'\exp\{i(m\theta-\omega_R t)\}\},\end{aligned}\right\} \tag{3.5}$$

where ω_R is $\mathrm{Re}\,\omega$ at critical conditions and $\chi(r)$ satisfies (2.21) with $\phi(r) = \frac{1}{2}\Omega_\omega - c$. Using (3.4) and (3.5) in (3.2) and performing the θ-averages yields

$$\left\{\frac{1}{2}\frac{d}{dt}+2\beta^{-1}\right\}A^2(t)\,I_1+A^2(t)\,I_2 m$$
$$-\tfrac{1}{8}\delta^2 m^2 A^2(t)\exp\{-2t\beta^{-1}\}\,I_3\int_0^t A^2(s)\exp\{2s\beta^{-1}\}\,ds = 0, \tag{3.6}$$

where

$$I_1 = \int_0^\infty \left[\left(\frac{m}{r}\right)^2(\chi_R^2+\chi_I^2)+(\chi_R')^2+(\chi_I')^2\right]r\,dr, \tag{3.7}$$

$$I_2 = \int_0^\infty \tfrac{1}{2}\Omega_w' f(\chi)\,r\,dr, \tag{3.8}$$

$$I_3 = \int_0^\infty f(\chi)\frac{d}{dr}[r^{-2}(\chi_R''\chi_I-\chi_I''\chi_R)+r^{-3}f(\chi)]\,r\,dr, \tag{3.9}$$

$$\chi = \chi_R+i\chi_I,\quad f(\chi) = \chi_R'\chi_I-\chi_I'\chi_R.$$

The time development of the perturbation amplitude $A(t)$ will follow from (3.6), after an expression for the spatial dependence $\chi(r)$ is obtained and (3.7)–(3.9) are evaluated.

4. Approximate spatial dependence of the wave disturbance

As far as the author is aware, no closed-form expression is known for a solution of (2.21) corresponding to a neutrally stable disturbance of an everywhere differentiable $\phi(r)$. There is at least one such solution for the Rayleigh equation analogous to (2.21) in Cartesian co-ordinates (see Drazin & Howard 1966; Michalke 1964), but along with other disadvantages, its use for the evaluation of (3.7)–(3.9) would restrict our analysis to a very specific wall profile $\Omega_w(r)$. Known solutions of (2.21) for continuous but only piecewise differentiable $\phi(r)$ (Michalke

& Timme 1967; Busse 1968) are also unsuitable for our purposes, since all the contributions to (3.9) occur at points where the integrand has non-integrable singularities. To avoid the remaining alternative of a numerical determination of χ, and thereby to keep the calculations both reasonably simple and somewhat insensitive to the precise form of Ω_w, we exploit an approximation scheme analogous to that used by Drazin & Howard (1962) for the Rayleigh equation.

It is now necessary to specify more carefully the properties of the class of smooth functions Ω_w we shall consider. The *indispensible* assumption is that Ω_w depends on a small parameter ϵ such that wherever $r/a - 1 = O(1)$ as $\epsilon \to 0$, the asymptotic expansion of Ω_w in powers of ϵ is a constant, i.e. Ω_w deviates from a constant only by terms such as $\exp\{-r/\epsilon\}$ or $r^{-1/\epsilon}$ which have asymptotic expansion zero. An additional *convenient* assumption is that

$$\phi(r) = \tfrac{1}{2}\Omega_w(r) - c \equiv \Phi(\epsilon^{-1}\ln(r/a)),$$

where Φ is a function (e.g. a hyperbolic tangent) which approaches a constant as its argument tends to $\pm\infty$. The usefulness of this second assumption will become apparent in the next paragraph, and its non-necessity is discussed later. For $r/a - 1 = O(1)$ the second group of terms in (2.21) are thus negligibly small, so we can approximate the solution by

$$\chi(r) = \left\{\begin{array}{ll}\phi(0)\,(r/a)^m, & r < a,\\ K(a/r)^m, & r > a,\end{array}\right\} \tag{4.1}$$

where we have chosen the value of ϕ at $r = 0$ for the normalization of χ ($\phi(0)$ is of course essentially $\phi(a-)$, and $\phi(\infty)$ is $\phi(a+)$). We must next obtain an approximation for χ in $r/a - 1 = o(1)$, then match these solutions *smoothly*; and completion of the matching produces an approximate determination of the eigenvalue c and constant K.

For the inner region $r/a - 1 = o(1)$, we first transform (2.21) to the variable $\eta = \epsilon^{-1}\ln r/a$, so that for the solution $\chi(r) \equiv \tilde{\chi}(\eta)$ we have

$$\Phi(\epsilon^{-2}\tilde{\chi}'' - m^2\tilde{\chi}) - (\epsilon^{-2}\Phi'' + 2\epsilon^{-1}\Phi')\,\tilde{\chi} = 0. \tag{4.2}$$

The reason for introducing the logarithmic transformation is to avoid difficulties of non-uniform validity in the subsequent perturbation expansion for $\tilde{\chi}$ which would arise if, e.g., the variable $\epsilon^{-1}(r/a - 1)$ has been used instead. The matching conditions for (4.2) are

$$\left.\begin{array}{lll}\tilde{\chi} \to \phi(0), & \tilde{\chi}' \to \epsilon m\phi(0) & \text{as} \quad \eta \to -\infty,\\ \tilde{\chi} \to K, & \tilde{\chi}' \to -\epsilon mK & \text{as} \quad \eta \to \infty.\end{array}\right\} \tag{4.3}$$

Again to avoid non-uniformity we seek solutions in the form

$$\tilde{\chi}_\pm(\eta) = \exp\{\mp\epsilon m\eta\}\,\psi_\pm(\eta) \quad \text{for} \quad n \gtrless 0, \tag{4.4}$$

and (4.2) and (4.3) become

$$\Phi(\psi''_\pm \mp 2m\epsilon\psi'_\pm) - (\Phi'' + 2\epsilon\Phi')\,\psi_\pm = 0, \tag{4.5}$$

$$\left.\begin{array}{l}\psi_- \to \phi(0) \equiv \Phi(-\infty), \quad \psi'_- \to 0 \quad \text{as} \quad \eta \to -\infty,\\ \psi_+ \to K, \quad \psi'_+ \to 0 \quad \text{as} \quad \eta \to \infty.\end{array}\right\} \tag{4.6}$$

If we expand the functions $\psi_\pm$ in series

$$\psi_\pm = \psi^0_\pm + \epsilon\psi^1_\pm + \epsilon^2\psi^2_\pm + \dots, \tag{4.7}$$

we can obtain the $\psi^n_\pm$ successively. It is clear that, if we had not specified the convenient dependence of $\phi(r)$ on the combination $\epsilon^{-1}\ln(r/a)$, then under the change of variable from r to η, $\phi(r)$ would become a function $F(\eta,\epsilon)$. The function F would have to be expanded in a series like (4.7), and the corresponding problems for $\psi^n_\pm$ would have additional terms from the $F^n(\eta)$; but we do not pursue this case here.

One matching procedure is to apply the conditions to $\psi_\pm$ as $\eta \to \pm\infty$, respectively, then use connexion formulae

$$\tilde{\chi}_+(0) = \tilde{\chi}_-(0), \quad \tilde{\chi}'_+(0) = \tilde{\chi}'_-(0)$$

to determine K and c. We follow instead an alternative procedure (Drazin & Howard 1962) of normalizing $\tilde{\chi}_\pm(\eta)$ separately as $\eta \to \pm\infty$, and requiring these two solutions to be linearly dependent (i.e. to be different representations of the same eigenfunction on $(-\infty,\infty)$). This device simplifies the calculations, because it allows us to specify the convenient value

$$K = \phi(\infty) \equiv \Phi(\infty), \tag{4.8}$$

and because the eigenvalue relation $\tilde{\chi}_+\tilde{\chi}'_- - \tilde{\chi}'_+\tilde{\chi}_- = 0$ may be evaluated at the point $\eta = 0$. In terms of the ψ's this condition is

$$\psi_+(0)\,\psi'_-(0) - \psi'_+(0)\,\psi_-(0) + 2m\epsilon\psi_+(0)\psi_-(0) = 0. \tag{4.9}$$

Carrying out this programme, we use (4.7) in (4.5) to obtain

$$[\Phi^2[\psi^0_\pm/\Phi]']' = 0, \tag{4.10}$$

$$[\Phi^2[\psi^1_\pm/\Phi]']' = -\Phi^3/[\psi^0_\pm]^{\mp 2m-1}[[\psi^0_\pm]^{\mp 2m}/\Phi^2]', \tag{4.11}$$

$$[\Phi^2[\psi^2_\pm/\Phi]']' = -\Phi^3/[\psi^1_\pm]^{\mp 2m-1}[[\psi^1_\pm]^{\mp 2m}/\Phi^2]'. \tag{4.12}$$

With (4.6) and (4.8), the solution of (4.10) is

$$\psi^0_\pm = \Phi(\eta). \tag{4.13}$$

Equations (4.1), (4.4), (4.8) and (4.13) give an adequate approximation to the spatial form of the eigenfunction for use in (3.7)–(3.9). However, (4.13) identically satisfies (4.9) to $O(\epsilon)$ terms, so it is necessary to find higher-order terms in (4.7) to obtain an approximation to c. Since effects of cylindrical geometry appear in the $O(\epsilon)$ terms of (4.5), it is apparent that curvature will affect the eigenvalue.

The solutions of (4.10) and (4.11) that satisfy (4.6) are

$$\psi^1_\pm = (1\pm m)\,\Phi(\eta)\int_{\pm\infty}^{\eta}[1-\Phi^{-2}(\eta_1)\,\Phi^2(\pm\infty)]\,d\eta_1, \tag{4.14}$$

$$\psi^2_\pm = (1\pm m)\,\Phi(\eta)\int_{\pm\infty}^{\eta} d\eta_1\Big\{(1\pm m)\int_{\pm\infty}^{\eta_1}[1-\Phi^{-2}(\eta_2)\,\Phi^2(\pm\infty)]\,d\eta_2 - (1\mp m)\,\Phi^{-2}(\eta_1)\int_{\pm\infty}^{\eta_1}[\Phi^2(\eta_2)-\Phi^2(\pm\infty)]\,d\eta_2\Big\}. \tag{4.15}$$

Now using (4.7) with (4.13)–(4.15) in (4.9), we find, after some straightforward calculations,

$$\epsilon[(m-1)\phi^2(0)+(m+1)\phi^2(\infty)]+\epsilon^2(m^2-1)\int_{-\infty}^{\infty}\{\Phi^{-2}(\eta_1)$$
$$\times[\Phi^2(\eta_1)-\phi^2(\infty)][\Phi^2(\eta_1)-\phi^2(0)]\}d\eta_1+O(\epsilon^3)=0. \quad (4.16)$$

If $\Omega_w(0) \neq \Omega_w(\infty)$ (the 'shear-layer' case in Drazin & Howard), then (4.16) gives the approximation

$$mc = \tfrac{1}{4}[(m+1)\Omega_w(\infty)+(m-1)\Omega_w(0)] \pm \tfrac{1}{4}i(m^2-1)^{\frac{1}{2}}[\Omega_w(\infty)-\Omega_w(0)]+O(\epsilon). \quad (4.17)$$

In the 'jet' case $\Omega_w(0) = \Omega_w(\infty)$ (a constant which can be taken zero by a different definition of the angular rotation of the co-ordinate system), (4.16) becomes

$$mc = \pm i\epsilon^{\frac{1}{2}}\left\{\tfrac{1}{2}m(m^2-1)\int_{-\infty}^{\infty}[\tfrac{1}{2}\Omega_w(\eta)]^2\,d\eta\right\}^{\frac{1}{2}}.$$

We shall not make use of this result, which is included to indicate the extension of Drazin & Howard's formula (2.10) to this problem.

5. The equation for the perturbation amplitude

We use the approximations for $\chi(r)$ developed in the preceding section for the evaluation of the largest contributions to (3.7)–(3.9). From (4.1) it is clear that *all* the contributions to (3.8) and (3.9) occur in the region $\eta = O(1)$ of non-vanishing perturbation Reynolds stress. Moreover, (3.7) is $O(\epsilon^{-1})$ because of the $O(\epsilon^{-2})$ term $(\chi'_R)^2$ integrated over the variable η. The important estimates we require are, for the region $\eta = O(1)$,

$$\chi_R(\eta) = \exp\{-m\epsilon|\eta|\}\,\Phi_R(\eta)+O(\epsilon), \quad (5.1)$$

$$\chi_I(\eta) = -c_I\exp\{-m\epsilon|\eta|\}+O(\epsilon), \quad (5.2)$$

$$\frac{d\chi_R}{dr} = \epsilon^{-1}r^{-1}\exp\{-m\epsilon|\eta|\}\,\Phi'_R(\eta)+O(1), \quad (5.3)$$

$$\frac{d^2\chi_R}{dr^2} = \epsilon^{-2}r^{-2}\exp\{-m\epsilon|\eta|\}\,\Phi''_R(\eta)+O(\epsilon^{-1}). \quad (5.4)$$

The derivative $d\psi_I/dr$ will be at most $O(1)$, so that for $\eta = O(1)$, the second term in $f(\chi)$ is dominated by the first, $\chi_I d\chi_R/dr$. Similarly, $\chi_I d^2\chi_R/dr^2$ overwhelms $\chi_R d^2\chi_I/dr^2$ and $r^{-1}f(\chi)$ in the integrand of (3.9).

Changing variables from r to η in (3.7), noting $dr = \epsilon r\,d\eta$, and using (5.3), we find

$$I_1 = \epsilon^{-1}\int_{-\infty}^{\infty}\exp\{-2m\epsilon|\eta|\}(\Phi'_R)^2\,d\eta+O(1). \quad (5.5)$$

Also, (5.2) and (5.3) in (3.8) give

$$I_2 = -\epsilon^{-1}c_I\int_{-\infty}^{\infty}\exp\{-2m\epsilon|\eta|\}(\Phi'_R)^2\,d\eta+O(1). \quad (5.6)$$

We integrate (3.9) by parts and then use (5.2) and (5.4) to obtain

$$I_3 = -(c_I)^2 \epsilon^{-3} a^{-4} \int_{-\infty}^{\infty} \exp\{-4m\epsilon|\eta|\} \exp\{-4\epsilon\eta\} (\Phi_R'')^2 d\eta + O(\epsilon^{-2}). \quad (5.7)$$

Now
$$\Phi_R(\eta) = \tfrac{1}{2}\Omega_w(ae^{\epsilon\eta}) - c_R$$

is a function with derivatives that must vanish rapidly as $|\eta|$ becomes large with respect to one. To within the accuracy of our approximations, we may set equal to one the exponential factors in the integrands of (5.5)–(5.7), since they differ from one only for $|\eta| = O(\epsilon^{-1})$ where the integrands are already negligibly small (since $m \geqslant 1$ the exponentials in the integrand of (5.7) are not large even for $\eta < 0$).

The most significant contributions to (3.7)–(3.9) are thus characterized by the integrals

$$A_1 \equiv \int_{-\infty}^{\infty} (\Phi_R')^2 d\eta, \quad A_2 \equiv \int_{-\infty}^{\infty} (\Phi_R'')^2 d\eta, \quad (5.8)$$

since, from (5.5)–(5.7),

$$\left.\begin{aligned} I_1 &= \epsilon^{-1}A_1 + O(1), \\ I_2 &= -c_I I_1 + O(1), \\ I_3 &= -(c_I)^2\epsilon^{-3}a^{-4}A_2 + O(\epsilon^{-2}). \end{aligned}\right\} \quad (5.9)$$

The shape properties of the wall velocity profile that are significant in this model are simply the integrals of the square of the first two profile derivatives. Moreover, inspection of (3.7) and (5.9) reveals that only the ratio $R = A_2/A_1$ influences the perturbation amplitude development.

We next consider a few simple examples, to indicate the magnitude of R. We observe that a can be picked so that $\Phi'(0)$ is non-zero, and that $\Phi'(0)$ may be normalized as $-\frac{1}{2}$, since any other value would only modify the definition of ϵ. Also, the limiting values $\Phi(-\infty) = \frac{1}{2} - c$ and $\Phi(\infty) = -c$ are appropriate for a wall angular velocity which is uniform for $r < a$ and zero for $r > a$. Another very important requirement of Φ is rapid approach to its limiting values as $\eta \to \pm\infty$, as discussed in §4.

(i) The function
$$\Phi = \tfrac{1}{2}[\tfrac{1}{2}(1 - \tanh 2\eta)] - c$$

is the analogue in our model for the profile considered by Schade (1964) and Stuart (1967) in a non-rotating fluid. The integrals in (5.8) are $A_1 = \frac{1}{6}$ and $A_2 = \frac{8}{15}$, so $R = 3{\cdot}2$.

(ii) The profile
$$\Phi = \tfrac{1}{2}[H(-\eta) + \tfrac{1}{2}(\operatorname{sgn}\eta)\exp\{-2|\eta|\}] - c$$

has the functional form of the solution for the thicker shear layer in Stewartson's (1957) model. Except for modifications introduced by the logarithmic transformation from r to η, it represents the averaged angular velocity in the fluid in the absence of interaction with the perturbation when Ω_w is a step function. The integral A_1 is $\frac{1}{8}$ and $R = 4$.

(iii) A class of profiles for which the required integrals are elementary is

$$\Phi(\eta) = \frac{1}{2}\left[1 - I_\alpha^{-1}\int_{-\infty}^{\eta I_\alpha} \exp\{-|t|^\alpha\}\,dt\right] - c, \tag{5.10}$$

where $\alpha > \frac{1}{2}$ and $I_\alpha \equiv \int_{-\infty}^{\infty} \exp\{-|t|^\alpha\}\,dt = 2\alpha^{-1}\Gamma(\alpha^{-1})$.

The integrals in (5.8) are

$$A_1 = 2^{-(2+\alpha^{-1})} \quad \text{and} \quad A_2 = 2^{-(2-\alpha^{-1})}\Gamma(\alpha^{-1})\,\Gamma(2-\alpha^{-1}),$$

so that

$$R = 2^{2\alpha^{-1}}\Gamma(\alpha^{-1})\,\Gamma(2-\alpha^{-1}) \quad (R = 2^{2\alpha^{-1}}(1-\alpha^{-1})\,\pi\operatorname{cosec}\pi\alpha^{-1} \quad \text{for} \quad \alpha > 1).$$

This ratio increases as α decreases to $\frac{1}{2}$ because of the cusp in the second derivative of Φ at $\eta = 0$, and it increases as α increases to ∞ because of the sharpening of the profile near $\eta = \pm I_\alpha^{-1}$. For $\alpha = 1$ this is of course example (i); for $\alpha = 2R$ is π; and rough calculations indicate the minimum value of R is about 3·0 near $\alpha = 1{\cdot}6$.

Other examples involving functions similar to those above have been evaluated, and the results affirm that $R = 4$ is a representative choice for the profile conditions under consideration.

With (5.9) and the definition $B(t) \equiv A^2(t)$, (3.6) becomes

$$\tfrac{1}{2}B'(t) - \omega_I B + KB\exp\{-2t\beta^{-1}\}\int_0^t B(s)\exp\{2s\beta^{-1}\}\,ds = 0, \tag{5.11}$$

where $$K \equiv \left(\frac{\delta}{\epsilon}\right)^2 \frac{m^2c_I^2R}{8a^4} \quad \text{and} \quad \omega_I = mc_I - 2\beta^{-1}$$

from (2.20). Equation (5.11) is equivalent to the second-order autonomous system

$$\frac{dB}{dt} = C, \quad B\frac{dC}{dt} = C^2 - 2\beta^{-1}BC + 4\beta^{-1}\omega_I B^2 - 2KB^3, \tag{5.12}$$

with (B, C) at $t = 0$ equal to $(B(0),\ 2\omega_I B(0))$. The origin in the B, C plane is a second-order singular point of (5.12), and its instability for $\omega_I > 0$ is presumed, in view of (5.11). The only other singular point of (5.12) is $(2\omega_I\beta^{-1}K^{-1}, 0)$, and for $\omega_I > 0$ a short calculation shows it is a stable spiral ($\omega_I > (4\beta)^{-1}$) or a stable node ($\omega_I < (4\beta)^{-1}$).

The predicted equilibrium squared amplitude $B_e = A_e^2$ toward which the perturbation tends therefore satisfies

$$\delta^2 A_e^2 = \epsilon^2\left(\frac{4a^2}{mc_I}\right)\left(\frac{\omega_I}{R\beta}\right). \tag{5.13}$$

Equation (5.13) is written in this form to indicate that $\delta = O(\epsilon)$ is a consequence of our perturbation solution. It is a straightforward matter to verify that our perturbation scheme is consistent, by re-examining the derivation of (5.13) with $\delta = O(\epsilon)$ *ab initio*. Furthermore, (5.13) emphasizes that δA_e, rather than δ or A_e individually, is the quantity of observational interest. The prediction of δA_e could be checked experimentally if e.g. the maximum radial velocity is measured in one of the vortices of the instability pattern, and (5.13) is used along with results from §§3 and 4 for the velocity solution. We note finally that, for the profile conditions $\Omega_w(0)-1 = \Omega_w(\infty) = 0$, (5.13), along with (4.17), (2.20), and $R = 4$, gives

$$(\delta A_e)^2 = \epsilon^2 \frac{64a^4}{(m^2-1)\beta}[\tfrac{1}{4}(m^2-1)^{\frac{1}{2}} - 2\beta^{-1}]. \tag{5.14}$$

It is worth mentioning that (5.13) is also the equilibrium point for the analogous perturbation amplitude equation when (2.13) for the averaged flow is solved without the time-derivative term (see discussion preceding (3.4)). This conclusion is not unexpected for our model, but it serves to substantiate the neglect by Stuart (1958) of similar terms the effects of which on his calculations could not be fully assessed. The time-derivative terms will of course modify the approach of the perturbation amplitude to A_e, as illustrated by the predicted overshooting of A_e as t evolves for $\omega_I > (4\beta)^{-1}$.

6. Discussion

We first want to indicate the use of (5.13) or (5.14) in the comparison of the theory with the actual evolution of shear-layer instabilities. Among the quantities of experimental interest are the distribution of perturbation Reynolds stress, the distorted averaged velocity profile, and the amplitude of the instability vortices relative to the averaged flow. There are unfortunately no suitable data with which we can directly compare our predictions, but we can use the best currently available data (from Hide & Titman) to obtain some predictions of a quotient Q. Q is the maximum speed in the equilibrium vortices outside the shear layer divided by the shear-layer velocity at $r = a$ in the absence of non-axisymmetric perturbations, i.e.

$$Q = \frac{\max\limits_{(r,\theta)} \omega H\delta\,[u^2+v^2]^{\frac{1}{2}}|_{t\to\infty}}{\omega H V(a)}. \tag{6.1}$$

With (4.1), (4.8) and (4.17), the function $\chi(r)$ outside the transition region is

$$\chi(r) = \frac{1}{4}\left(\frac{r}{a}\right)^{\pm m}\left[\left(\pm 1 + \frac{1}{m}\right) - \frac{i}{m}(m^2-1)^{\frac{1}{2}}\right] \quad (r \lessgtr a).$$

From (3.5) the perturbation radial velocity is

$$u(r,\theta,t) = -\frac{mA(t)}{8a}\,\mathrm{Re}\left\{\left(\frac{r}{a}\right)^{\pm m-1}\left[i\left(\pm 1+\frac{1}{m}\right) + \frac{(m^2-1)^{\frac{1}{2}}}{m}\right]\exp\{i(m\theta - \omega_R t)\}\right\}$$
$$(r \lessgtr a),$$

and the azimuthal velocity v differs from u by a phase of $\pm\frac{1}{2}\pi$ for $r \lessgtr a$. Hence, according to our approximations,

$$u(r,\theta,t\to\infty) = -\frac{A_e 2^{\frac{1}{2}}}{8a}(m^2 \pm m)^{\frac{1}{2}}\left(\frac{r}{a}\right)^{\pm m-1}\begin{pmatrix}\cos\tau\\ \sin\tau\end{pmatrix} \quad (r \lessgtr a),$$

where
$$\tau = m\theta - \omega_R t + \tan^{-1}\left(\frac{m+1}{m-1}\right)^{\frac{1}{2}},$$

and, using the similar expression for $v(r,\theta,t\to\infty)$, we have

$$\max_{(r,\theta)}[u^2+v^2]^{\frac{1}{2}}|_{t\to\infty} = \frac{A_e 2^{\frac{1}{2}}}{8a}(m^2+m)^{\frac{1}{2}}. \tag{6.2}$$

Hide & Titman's shear layer was driven by a co-axial disk, which is equivalent to a wall velocity $V_w = rH(a-r)$. From calculations by Stewartson (1957), it is known that the thickest part of the shear layer from such a wall velocity has the form

$$V(r) = \tfrac{1}{2}rH(a-r)+\tfrac{1}{4}a[\operatorname{sgn}(r-a)]\exp\{-2^{\frac{1}{2}}E^{-\frac{1}{4}}|r-a|\}. \tag{6.3}$$

To apply the results from our model, we must identify the analogous wall angular velocity Ω_A which in the absence of perturbations would produce the velocity $V(r)$ in (6.3). From (3.4) it is clear that

$$\Omega_A(r) = H(a-r)+\tfrac{1}{2}[\operatorname{sgn}(r-a)]\exp\{-2^{\frac{1}{2}}E^{-\frac{1}{4}}|r-a|\}. \tag{6.4}$$

Moreover, we may identify ϵ from the definition of the function $\Phi(\eta)$ and its normalization,

$$\left.\begin{aligned} \frac{1}{2}\frac{d\Omega_A}{dr}\bigg|_{r=a} = \frac{1}{\epsilon a}\frac{d\Phi}{d\eta}\bigg|_{\eta=0} = -\frac{1}{2\epsilon a},\\ \epsilon = |a\Omega_A'(a)|^{-1} = 2^{\frac{1}{2}}a^{-1}E^{\frac{1}{4}}. \end{aligned}\right\} \tag{6.5}$$

As anticipated, the value for ϵ in (6.5) is just outside the range for which our theory was derived; nevertheless, we use our formula to exploit Hide & Titman's data.

Using (5.14), (6.2), (6.3) and (6.5) in (6.1), we obtain the following expression for Q in terms of the growth rate ω_I and other parameters:

$$\left.\begin{aligned} Q = \frac{8}{a}\left(\frac{m}{m-1}\right)^{\frac{1}{2}}(\omega_I E R_0^{-1})^{\frac{1}{2}},\\ \omega_I = \tfrac{1}{4}(m^2-1)^{\frac{1}{2}}-2E^{\frac{1}{2}}R_0^{-1}. \end{aligned}\right\} \tag{6.6}$$

We observe that the growth rate increases nearly linearly with m, while the explicit dependence of Q on m is slight. The growth rate is independent of a, and Q is inversely proportional to a because the axisymmetric shear flow at $r = a$ increases with disk radius. The variation of Q with E and R_0 is similar to, but not identical with, the variation of ω_I with these parameters.

With data from Hide & Titman's table A 1 and the relationships

$$E = E_{HT}a^2, \quad R_0 \cong \epsilon_{HT}, \quad a = \frac{a_{HT}(\mathrm{cm})}{8(\mathrm{cm})}$$

connecting Hide & Titman's (subscripted) parameters with ours, we can evaluate (6.6). The results in table 1 show that Q decreases slightly as a increases, as expected, and that Q has very little dependence on m in spite of the strong variation of ω_I with m. Table 1 indicates that the 'strength' of the instability vortices is approximately one quarter of the speed of the basic flow, and this fraction is representative over a significant range of onset conditions.

The information obtained from approximations derived in §4 for the linearly unstable solutions represents another contribution of this paper. Equation (4.17) for the phase speed and growth rate of the unstable waves is independent of details of the profile shape and relies *only* on the large velocity derivative in, or the thinness of, the shear layer. It is easy to show that formulae (5.3) and (5.7) of Busse for particular shear-layer profiles reduce to our results, if Busse's parameter $\gamma \equiv (r_1/r_2)^2$ is set equal to $1-\epsilon$ and ϵ is nearly zero.

a_{HT}(cm)	m	$E_{TH} \times 10^{+5}$	$\epsilon_{HT} \times 10^{+2}$	ω_I	Q
2·50	2	44·6	21·8	0·37	0·30
	2	19·2	12·5	0·36	0·26
	3	14·0	10·4	0·63	0·28
	3	11·8	8·2	0·62	0·29
3·75	2	19·3	12·7	0·33	0·25
	3	13·0	10·9	0·60	0·26
	3	8·85	8·3	0·60	0·24
	4	5·32	6·3	0·86	0·22
5·00	3	7·46	7·3	0·55	0·23
	4	3·61	5·3	0·83	0·22
	5	2·94	4·5	1·07	0·23
6·25	4	4·71	5·9	0·79	0·23
	5	3·18	4·6	1·03	0·24
	5	2·37	3·9	1·02	0·22
7·25	4	4·61	5·8	0·81	0·23
	5	2·25	3·9	1·06	0·22
	6	1·62	3·2	1·31	0·22

TABLE 1. Data in columns 1–4 from table A 1 of Hide & Titman (1967); columns 5 and 6 from (6.6).

Moreover, (4.17) shows that the wavenumber $m = 1$ is stable under our assumptions, in accord with Busse's examples and with comments by Michalke & Timme (1967) (the latter, incidentally, show by their example (4.1) that discontinuity of $V(r)$ can destabilize $m = 1$). The result $c_R = \frac{1}{4}(1 - 1/m)$, which follows from (4.17) with $\Omega_w(0) = 1$ and $\Omega_w(\infty) = 0$, shows qualitatively the same dependence of wave speed on wavenumber as observations by Hide & Titman (their figure 10). Finally, we mention that the principal geometric curvature effects have been retained in our approximations for the linear wave speeds and growth rates, unlike the comparison by Busse (his figure 3), in which curvature effects are included in an entirely *ad hoc* manner.

Certain features of the observations by Hide & Titman have not been adequately described by the linear instability theory we have assumed. One issue concerns the instability onset condition of $R_0 = O(E^{\frac{1}{2}})$, for which Hide & Titman suggest instead the exponent $\frac{4}{7}$, while Busse gives an exponent $\frac{3}{4}$. Part of the difference between $\frac{1}{2}$ and $\frac{4}{7}$ may be due to the parameter definitions of Hide & Titman. Busse's condition is actually $R_0 = O[E^{\frac{1}{2}}(1-\gamma)]$, with $(1-\gamma) = O(E^{\frac{1}{4}})$ for the most unstable waves. The linear theory of §4 has, of course, the same property, that the most unstable wavenumber is comparable to the shear-layer width. Hide (1973, private communication) indicated that there is no firm experimental support for Busse's contention of the exponent $\frac{3}{4}$. In any case, additional experiments would be highly desirable to confirm the onset condition and to determine the growth rate as a function of wavenumber. Other defects of the linear theory include the form of the predicted shape of the unstable wave and the failure to produce the observed onset wavenumber m. The latter difficulty might be resolved with a numerical solution of the perturbation equation with lateral dissipation included, as is appropriate in the case $\epsilon = O(E^{\frac{1}{4}})$. A simple model of this effect is obtained by adding a term $-m^2\mu\nabla^2 p^0$ to the right side of (2.18), and the most unstable wavenumber is then $m_0 = [1+(1/(4\mu)^2]^{\frac{1}{2}}$. This mechanism is presumably at work to select the wavenumber at instability onset.

Apart from these deficiencies of the linear instability theory, the most serious limitation on the results of our analysis is the restriction $\epsilon \gg E^{\frac{1}{4}}$. Although this condition is violated by laboratory experiments with discontinuous wall velocities, it may be relevant to some geophysical situations; and it is a reasonable approximation, if the smoothing mechanism of the averaged flow is not dynamically significant to the instability. The case $\epsilon = O(E^{\frac{1}{4}})$ was not pursued, to consider a simple theory without the lateral dissipation mechanism. The following modifications and extensions of the theory have been made through some additional calculations: allowing smooth variation of the function F (in §4) with η; adding side walls to the flow domain; and including the lateral dissipation term in (2.13).

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